TORIC RESIDUE MIRROR CONJECTURE FOR CALABI-YAU COMPLETE INTERSECTIONS

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0. Introduction

The toric residue mirror conjecture of Batyrev and Materov [2, 3] expresses a toric residue as a power series whose coefficients are certain integrals over moduli spaces. This conjecture for Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties was proved independently by Szenes and Vergne [10] and Borisov [5]. We build on the work of these authors to generalize the residue mirror map to not necessarily reflexive polytopes. Using this generalization we prove the toric residue mirror conjecture for Calabi-Yau complete intersections in Gorenstein toric Fano varieties [3].

We start by introducing notation and explaining the main idea of the generalization. We work over the field $K = \mathbb{Q}$. Let $\overline{M} \simeq \mathbb{Z}^d$, let $\Delta \subset \overline{M}_K$ be a d-dimensional lattice polytope, and let $\mathcal{T}$ be a coherent triangulation of Δ, defined by a convex piecewise linear integral function on Δ. All lattice points in Δ are assumed to be vertices of the simplices in $\mathcal{T}$. We place Δ in $M_K = (\overline{M} \times \mathbb{Z})_K$ as $\Delta \times \{1\}$ and let $C_\Delta \subset M_K$ be the cone over Δ with vertex 0. Then $\mathcal{T}$ defines a subdivision of C_Δ into a fan Σ.

The idea of the toric residue mirror conjecture is to relate the semigroup ring $S_\Delta = K[C_\Delta \cap M]$ to the cohomology of the fan Σ. Let $I_\Delta \subset S_\Delta$ be the ideal generated by monomials t^m where $m \in M$ lies in the interior of C_Δ. Given general elements $f_0, \ldots, f_d \in S^1_\Delta$ (the superscript denotes the degree), we can construct the toric residue map [9]:

$$Res_{(f_0,\ldots,f_d)} : (I_\Delta/(f_0,\ldots,f_d)I_\Delta)^{d+1} \xrightarrow{\sim} K.$$

Following [2], we choose a special set of f_i constructed from a single $f \in S_\Delta$ by partial differentiation.

On the cohomology side, the Poincaré dual of the cohomology $H(\Sigma)$ is the cohomology with compact support $H(\Sigma, \partial\Sigma)$ [1]. In the top degree we have the evaluation map

$$\langle \cdot \rangle_\Sigma : H^{d+1}(\Sigma, \partial\Sigma) \xrightarrow{\sim} K.$$

The residue mirror map takes I^{d+1}_Δ into $H^{d+1}(\Sigma, \partial\Sigma)$ so that composition with the evaluation map gives the toric residue.

The toric residue mirror conjecture of Batyrev and Materov [2, 3] is a special case of the above formulation. If Δ is reflexive, it has only one lattice point 0 in its interior. Assume that every maximal simplex in $\mathcal{T}$ has 0 as a vertex. Then the projection $q : M_K \to \overline{M}_K$ maps the fan Σ to a complete fan $\overline{\Sigma}$ in $\overline{M}_K$. (Geometrically, the toric variety of Σ is the total space of a line bundle over the toric variety of $\overline{\Sigma}$.) The cohomology spaces of the two fans are isomorphic, hence we can express the toric residue in terms of the cohomology of $\overline{\Sigma}$.

In the complete intersection case we use the Cayley trick [3] to construct a polytope $\tilde{\Delta} \subset M_K = (\overline{M} \times \mathbb{Z}^r)_K$ and a fan Σ subdividing $C_{\tilde{\Delta}}$. The projection $q : M_K \to \overline{M}_K$ again maps Σ to a complete fan $\overline{\Sigma}$. (The geometry here is that the toric variety of Σ is the total space of a rank r vector bundle over the toric variety of $\overline{\Sigma}$.) Thus, we can express the toric residue in terms of the cohomology of $\overline{\Sigma}$.

In the complete intersection case the ring $S_{\tilde{\Delta}}$ is graded by $\mathbb{Z}^r_{\geq 0}$. Restricting the toric residue to a homogeneous component of $I_{\tilde{\Delta}}$ defines the mixed toric residue. We also prove a conjecture in [3] relating the mixed residues with mixed volumes of polytopes.

In the proofs we follow the algebraic approach of Borisov [5], but we replace the higher Stanley-Reisner rings with Jeffrey-Kirwan residues as in [10].

Notation. Given a lattice $M \simeq \mathbb{Z}^d$, we denote $M_K = M \otimes K$ and the dual lattice $N = M^* = Hom(M, \mathbb{Z})$. For $u \in M$ and $w \in N$, we let the pairing be $(w, u) \in K$. Given a homomorphism $q : M \to M'$ of lattices, we denote the scalar extension $M_K \to M'_K$ also by q.

1. Cohomology

We recall the equivariant definition of the cohomology of Σ (which is the cohomology of the associated toric variety) [6, 1].

Let $\mathcal{A}(\Sigma)$ be the ring of K-valued conewise polynomial functions on Σ, graded by degree. The cohomology $H(\Sigma)$ is defined as the quotient $\mathcal{A}(\Sigma)/I$, where I is the ideal generated by global linear functions.

One can recover the Stanley-Reisner description of cohomology as follows. Let $v_1, \ldots, v_n$ be the primitive generators of Σ (the first lattice points on the 1-dimensional cones of Σ), and let $\chi_i \in \mathcal{A}^1(\Sigma)$ be the conewise linear functions defined by

$$\chi_i(v_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol. Then χ_i for $i = 1, \ldots, n$ generate the ring $\mathcal{A}(\Sigma)$, with relations generated by monomials $\prod_{i \in I} \chi_i$, where $\{v_i\}_{i \in I}$ do not lie in one cone of Σ. To obtain the cohomology, we add the linear relations

$$\sum_{i=1}^{n} (w, v_i)\chi_i = 0$$

for all $w \in N = \mathrm{Hom}(M, \mathbb{Z})$.

Let $\mathcal{A}(\Sigma, \partial\Sigma)$ be the ideal in $\mathcal{A}(\Sigma)$ of functions vanishing on the boundary of Σ, and let $H(\Sigma, \partial\Sigma)$ be the quotient $\mathcal{A}(\Sigma, \partial\Sigma)/I\mathcal{A}(\Sigma, \partial\Sigma)$, where I is the ideal above. It is proved in [1] that multiplication of functions induces a non-degenerate bilinear pairing

$$H^k(\Sigma) \times H^{d+1-k}(\Sigma, \partial\Sigma) \to H^{d+1}(\Sigma, \partial\Sigma) \simeq K.$$

The isomorphism $H^{d+1}(\Sigma, \partial\Sigma) \simeq K$ can be defined as follows [6]. For $\sigma \in \Sigma$ a maximal cone, define $\Phi_\sigma = \prod_{v_i \in \sigma} \chi_i|_\sigma$, where $|_\sigma$ means that we consider Φ_σ as a global polynomial function on M_K whose restriction to σ is the product of χ_i. Let $Vol(\sigma)$ be the volume of the parallelotype generated by $v_i \in \sigma$. Equivalently, it is the index of the lattice generated

by $v_i \in \sigma$ in M. Now if $f \in \mathcal{A}^{d+1}(\Sigma, \partial\Sigma)$, consider the rational function

$$\langle f\rangle_\Sigma = \sum_{\sigma \in \Sigma^{d+1}} \frac{f|_\sigma}{\Phi_\sigma Vol(\sigma)}.$$

By Brion [6] the poles of this rational function cancel out, so that $\langle f\rangle_\Sigma$ is a constant, thus defining an isomorphism

$$\langle\cdot\rangle_\Sigma : H^{d+1}(\Sigma, \partial\Sigma) \xrightarrow{\sim} K.$$

We wish to give another description of the evaluation map using Jeffrey-Kirwan residues [7, 10]. The method works best for complete fans, so let us choose a completion $\hat{\Sigma}$ of Σ by adding a ray $K_{\geq 0}v_0$ for some $v_0 \in M$ such that $-v_0$ lies in the interior of C_Δ:

$$\hat{\Sigma} = \Sigma \cup \{K_{\geq 0}v_0 + \tau | \tau \in \partial\Sigma\}.$$

We have an embedding $H(\Sigma, \partial\Sigma) \subset H(\hat{\Sigma})$ defined by extending a function $f \in \mathcal{A}(\Sigma, \partial\Sigma)$ by zero outside the support of Σ. The evaluation map on $H(\hat{\Sigma})$ induces the evaluation map on $H(\Sigma, \partial\Sigma)$.

Let $\hat{\pi} : \mathbb{Z}^{n+1} \to M$ be the $\mathbb{Z}$-linear map $e_i \mapsto v_i$ for $e_0, \ldots, e_n$ the standard basis of $\mathbb{Z}^{n+1}$. The kernel of $\hat{\pi}$ is $R(\hat{\Sigma})$, the group of relations among v_i. We also let x_i for $i = 0, \ldots, n$ be the standard coordinate functions on K^{n+1}. Given a polynomial function $f(x_0, \ldots, x_n)$, we will consider its restriction to $R(\hat{\Sigma})_K \subset K^{n+1}$.

Let Q be the vector space of K-valued rational functions on $R(\hat{\Sigma})_K$ with poles lying along the hyperplanes defined by $x_i = 0$. Any element $g \in Q$ of degree $-(n-d) = -\dim R(\hat{\Sigma})_K$ can be written as a linear combination of basic fractions $(\prod_{i\in I} x^i)^{-1}$, where the images of $\{x_i\}_{i\in I}$ form a basis of the dual vector space $R(\hat{\Sigma})_K^*$, and degenerate fractions where the linear forms in the denominator do not span the dual.

The Jeffrey-Kirwan residue according to Brion and Vergne [7, 10] is a linear map

$$\langle\cdot\rangle_{JK(\hat{\Sigma})} : Q^{-(n-d)} \to K,$$

defined on the degenerate fractions to be zero and on the basic fractions:

$$\langle \frac{1}{\prod_{i\in I} x^i}\rangle_{JK(\hat{\Sigma})} = \begin{cases} \frac{1}{Vol(\sigma)} & \text{if } \{v_i\}_{i\notin I} \text{ generate a cone } \sigma \in \hat{\Sigma}, \\ 0 & \text{otherwise.} \end{cases}$$

The evaluation map on $H^{d+1}(\hat{\Sigma})$ can be given in terms of the Jeffrey-Kirwan residue as follows. Let $f(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree $d+1$. Then

$$\langle f(\chi_0, \ldots, \chi_n)\rangle_{\hat{\Sigma}} = \langle \frac{f(x_0, \ldots, x_n)}{x^{\mathbf{1}}}\rangle_{JK(\hat{\Sigma})},$$

where $x^{\mathbf{1}} = x_0 x_1 \cdots x_n$.

Lemma 1.1. *Let $x^m = x_0^{m_0} \cdots x_n^{m_n} \in K[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a monomial of degree $-(n-d)$. If $\{v_i\}_{m_i \geq 0}$ do not lie in one cone $\sigma \in \hat{\Sigma}$ then*

$$\langle x^m\rangle_{JK(\hat{\Sigma})} = 0.$$

Proof. Write $x^m = x^{m^+}/x^{m^-}$, where $m^+ = \max(m,0)$ and $m^- = \max(-m,0)$. Then x^m can be expressed as a linear combination of degenerate fractions and basic fractions of the form x^{-l}, where $0 \leq l_i \leq m_i^-$ for $i = 0,\dots,n$. If $\{v_i\}_{m_i \geq 0}$ do not lie in one cone then for no such l can $\{v_j\}_{l_j=0}$ generate a cone in $\hat{\Sigma}$. □

Szenes and Vergne [10] expressed the previous lemma in terms of the Mori cone as follows. Call an element $L \in \mathcal{A}^1(\Sigma)$ ample if it is strictly convex, and a fan quasi-projective if there exists an ample element. By the assumption that the triangulation $\mathcal{T}$ is coherent, the fan Σ is quasi-projective. The classes of ample elements form an open set in $H^1(\Sigma)$ whose closure is called the ample cone. The dual of the ample cone in $H^1(\Sigma)^* = R(\Sigma)_K$ is the Mori cone of Σ. Here $R(\Sigma) = \ker(\pi : \mathbb{Z}^n \to M)$, $\pi(e_i) = v_i$ for $e_1,\dots,e_n$ the standard basis of $\mathbb{Z}^n$. We denote the lattice points in the Mori cone by $R(\Sigma)_{\text{eff}}$.

For the following we need to observe that if $\beta = (\beta_1,\dots,\beta_n) \in R(\Sigma)$ is such that $\{v_i\}_{\beta_i<0}$ lie in one cone $\sigma \in \Sigma$, then $\beta \in R(\Sigma)_{\text{eff}}$. Indeed, any ample L can be modified by a global linear function so that it vanishes on σ and is strictly positive outside of σ, hence its pairing with β is non-negative.

Lemma 1.2. *Let $x^m \in K[x_1,\dots,x_n]$ be a monomial of degree $d+1$, and let $\beta \in R(\Sigma)$. If $\beta \notin R(\Sigma)_{\text{eff}}$ then*

$$\langle \frac{x^{m-\beta}}{x^{\mathbf{1}}} \rangle_{JK(\hat{\Sigma})} = 0.$$

Proof. Since $m_i \geq 0$, we have

$$\{i\}_{\beta_i<0} \subset \{i\}_{m_i-\beta_i-1\geq 0}.$$

It follows from the previous lemma that the Jeffrey-Kirwan residue is nonzero only if $\{v_i\}_{\beta_i<0}$ is a subset of a cone $\sigma \in \Sigma$, hence $\beta \in R(\Sigma)_{\text{eff}}$. □

2. Toric residues

We recall the definition of toric residues [9, 8, 2].

Recall that we defined S_Δ to be the semigroup ring of $C_\Delta \cap M$ and $I_\Delta \subset S_\Delta$ the ideal generated by monomials t^m where m lies in the interior of C_Δ. The ring S_Δ is Cohen-Macaulay with dualizing module I_Δ. Given a regular sequence $f_0,\dots,f_d \in S^1_\Delta$, the quotient $S_\Delta/(f_0,\dots,f_d)$ is again Cohen-Macaulay with dualizing module $I_\Delta/(f_0,\dots,f_d)I_\Delta$. It follows that there exists an isomorphism

$$(I_\Delta/(f_0,\dots,f_d)I_\Delta))^{d+1} \xrightarrow{\sim} K.$$

This isomorphism, normalized so that the Jacobian of $f_0,\dots,f_d$ maps to $Vol(\Delta)$ is called the toric residue $Res_{(f_0,\dots,f_d)}$. Here $Vol(\Delta)$ is $d!$ times the d-dimensional volume of Δ ($Vol(\Delta) = \sum_{\sigma\in\Sigma} Vol(\sigma)$). The Jacobian is defined by choosing a basis u_i for M, letting $t_i = t^{u_i}$, and considering $S_\Delta \subset K[t_0^{\pm 1},\dots,t_d^{\pm 1}]$. Then

$$Jac_{(f_0,\dots,f_d)} = \det(t_j \frac{\partial f_i}{\partial t_j})_{i,j}.$$

The Jacobian lies in I_Δ and it does not depend on the chosen basis.

Following Batyrev and Materov [2], we consider a regular sequence $f_0, \dots, f_d$, where

$$f_i = t_i \frac{\partial f}{\partial t_i}, \quad i = 0, \dots, d$$

and

$$f = \sum_{i=1}^{n} a_i t^{v_i},$$

with a_i parameters in K. The Jacobian now becomes the Hessian of f:

$$H_f = \det(t_i \frac{\partial}{\partial t_i} t_j \frac{\partial}{\partial t_j} f)_{i,j=0,\dots,d}.$$

Since $t_i \frac{\partial}{\partial t_i} t^{v_k} = (w_i, v_k) t^{v_k}$, where $w_0, \dots, w_d$ is the basis of N dual to $u_0, \dots, u_d$, we can write the Hessian as

$$H_f = \det(\sum_{k=1}^{n} (w_i, v_k)(w_j, v_k) a_k t^{v_k})_{i,j=0,\dots,d}.$$

By [8] the Hessian can also be expanded as

$$H_f = \sum_{J \subset \{1,\dots,n\}; |J|=d+1} V(J)^2 \prod_{i \in J} a_i t^{v_i},$$

where $V(J)$ is the volume of the cone generated by $\{v_i\}_{i \in J}$ (note that this cone may not be a cone in Σ). Since $V(J) \neq 0$ only if $\sum_{i \in J} v_i$ lies in the interior of C_Δ, it follows that $H_f \in I_\Delta^{d+1}$. When $f_0, \dots, f_d$ forms a regular sequence, the Hessian H_f does not lie in $(f_0, \dots, f_d) I_\Delta$, hence the normalization $Res_{a_1,\dots,a_n}(H_f) = Vol(\Delta)$ defines a unique linear map

$$Res_{a_1,\dots,a_n} : (I_\Delta / (f_0, \dots, f_d) I_\Delta))^{d+1} \xrightarrow{\sim} K.$$

3. The residue mirror map

Let $\pi : \mathbb{Z}^n \to M$ be the $\mathbb{Z}$-linear map $e_i \mapsto v_i$ for $i = 1, \dots, n$. We define the residue mirror map on monomials $t^l \in I_\Delta^{d+1}$ by

$$RM : t^l \mapsto \sum_{m \in \pi^{-1}(l)} \langle \left(\frac{x}{a}\right)^m \frac{1}{x^{\mathbf{1}}} \rangle_{JK(\hat{\Sigma})}$$

and extend linearly. Here $x^{\mathbf{1}} = x_0 x_1 \cdots x_n$,

$$\left(\frac{x}{a}\right)^m = \prod_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^{m_i},$$

and the sum on the right hand side is considered as a formal sum over Laurent monomials in a_i. Note that such sums do not form a ring, however multiplication of a formal sum with a Laurent polynomial in a_i is well-defined.

If $l = \pi(m_0)$ for some $m_0 \in \mathbb{Z}^n_{\geq 0}$, then using Lemma 1.2, we have

$$RM : t^l \mapsto \sum_{\beta \in R(\Sigma)} \langle \left(\frac{x}{a}\right)^{m_0 - \beta} \frac{1}{x^{\mathbf{1}}} \rangle_{JK(\hat{\Sigma})} = \sum_{\beta \in R(\Sigma)_{\text{eff}}} \langle \left(\frac{x}{a}\right)^{m_0 - \beta} \frac{1}{x^{\mathbf{1}}} \rangle_{JK(\hat{\Sigma})}.$$

Here the formal sum is a Laurent series in a_i with support lying in the cone $-m_0+R(\Sigma)_{\text{eff}}$. We denote by $K[[a_1,\dots,a_n]]$ the ring of such Laurent series (over all $m_0\in\mathbb{Z}^n$).

The following two lemmas and their proofs are only slight modifications of the ones in [5].

Lemma 3.1. *The map RM takes the subspace $((f_0,\dots,f_d)I_\Delta)^{d+1}$ to zero.*

Proof. Consider the linear map from S_Δ to the space of formal sums defined on monomials

$$t^l\mapsto\sum_{m\in\pi^{-1}(l)}\left(\frac{x}{a}\right)^m.$$

This is a map of $K[x_1,\dots,x_n]$ modules if we let x_i act on S_Δ by multiplication with $a_it^{v_i}$, and on the formal sums by multiplication with x_i.

A linear combination g of $f_0,\dots,f_d$ is given by

$$g=\sum_{i=1}^n(w,v_i)a_it^{v_i}$$

for some $w\in N_K$. Thus, multiplication with g in S_Δ corresponds to multiplication with $\sum_{i=1}^n(w,v_i)x_i$ in the module of formal sums. Now $R(\hat\Sigma)_K\subset K^{n+1}$ is defined by linear equations

$$\sum_{i=1}^n(w,v_i)x_i+(w,v_0)x_0=0.$$

Hence it suffices to show that

$$\langle x_0\left(\frac{x}{a}\right)^m\frac{1}{x^{\mathbf{1}}}\rangle_{JK(\hat\Sigma)}=0$$

for any $m\in\mathbb{Z}^n$ such that $\pi(m)$ lies in the interior of C_Δ. By Lemma 1.1, this residue is nonzero only if $\{v_0\}\cup\{v_i\}_{m_i\geq0}$ lie in a single cone of $\hat\Sigma$; in other words, $\{v_i\}_{m_i\geq0}$ lie in a cone on the boundary of C_Δ. Since $\pi(m)\in Int(C_\Delta)$, this cannot happen. $\square$

For later use we generalize the situation slightly. Let

$$f_\gamma=\sum_{i=1}^n a_i\gamma_it^{v_i},$$

where $\gamma_i>0$ are defined by a $w_\gamma\in N_K$:

$$(w_\gamma,v_i)=\frac{1}{\gamma_i},\quad i=1,\dots,n.$$

Let H_{f_γ} be the Hessian of f_γ, and consider the residue mirror map RM applied to H_{f_γ} (the map RM is not changed by γ).

Lemma 3.2. *We have*

$$RM(H_{f_\gamma})=\sum_{\sigma\in\Sigma^{d+1}}Vol(\sigma)\prod_{v_i\in\sigma}\gamma_i.$$

In particular, when $\gamma=\mathbf{1}$,

$$RM(H_f)=Vol(\Delta).$$

Proof. We follow closely the proof of Borisov [5].

The Hessian H_{f_γ} has an expression

$$H_{f_\gamma} = \sum_{J\subset\{1,\dots,n\};|J|=d+1} V(J)^2 \prod_{i\in J} a_i\gamma_i t^{v_i}.$$

We lift v_i to $e_i \in \mathbb{Z}^n$, then

$$RM(H_{f_\gamma}) = \sum_{J\subset\{1,\dots,n\};|J|=d+1} V(J)^2 \sum_{\beta\in R(\Sigma)_{\mathrm{eff}}} \langle x^J\gamma^J \left(\frac{x}{a}\right)^{-\beta} \frac{1}{x^{\mathbf{1}}}\rangle_{JK(\hat{\Sigma})},$$

where we write $x^J = \prod_{i\in J} x_i$ and similarly for γ^J. When $\beta = 0$, we have

$$\langle \frac{x^J}{x^{\mathbf{1}}}\rangle_{JK(\hat{\Sigma})} = \begin{cases} \frac{1}{V(J)} & \text{if } \{v_i\}_{i\in J} \text{ generate a cone } \sigma\in\Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the contribution from $\beta = 0$ to $RM(H_{f_\gamma})$ is

$$\sum_{\sigma\in\Sigma} Vol(\sigma) \prod_{v_i\in\sigma} \gamma_i,$$

and it remains to show that the contribution from any $\beta \neq 0$ is zero.

Fix $\beta \neq 0$ and consider

$$H_{f_\gamma} = \det A = \det(\sum_{k=1}^{n} (w_i, v_k)(w_j, v_k) a_k\gamma_k t^{v_k})_{i,j=0,\dots,d},$$

where $w_0, \dots, w_d$ is a basis of N. Since we want to prove the vanishing of the contribution from β to $RM(H_{f_\gamma})$, we are allowed to change H_{f_γ} by a nonzero constant, so we may assume $\{w_j\}$ to be a basis of N_K instead of N. We choose the basis so that $w_0 = w_\gamma$ and $(w_j, v_0) = 0$ for $j = 1, \dots, d$. Then the first row of the matrix A with index $i = 0$ has jth entry

$$\sum_{k=1}^{n} (w_j, v_k) a_k t^{v_k}.$$

Note that $\sum_{k=1}^{n}(w_j, v_k)x_k + (w_j, v_0)x_0$ restricts to zero on $R(\hat{\Sigma})_K$. Since for $j = 1, \dots, d$, $(w_j, v_0) = 0$, we may set the entries $A_{0,j}$ for $j \neq 0$ to zero. From the entry $j = 0$ we get a factor of x_0.

Let $A_{0,0}$ be the minor of the matrix A obtained by removing the first row and the first column. Similarly to the case of A, we have:

$$A_{0,0} = \det(\sum_{k=1}^{n} (w_i, v_k)(w_j, v_k) a_k\gamma_k t^{v_k})_{i,j=1,\dots,d} = \sum_{J\subset\{1,\dots,n\};|J|=d} V(J)^2 \prod_{i\in J} a_i\gamma_i t^{v_i},$$

where now $V(J)$ is the d-dimensional volume of the cone generated by $\{v_i\}_{i\in J}$. This volume is computed by projecting from v_0 and using the volume form determined by the basis $w_1, \dots, w_d$.

By the above discussion, disregarding the nonzero constants, we have to show that

$$\sum_{J\subset\{1,\ldots,n\};|J|=d} V(J)^2\gamma^J\langle x_0\frac{x^J}{x^{\beta+\mathbf{1}}}\rangle_{JK(\hat{\Sigma})}=0. \tag{1}$$

Here $\beta+\mathbf{1}=(1,\beta_1+1,\ldots,\beta_n+1)$. By Lemma 1.1, the Jeffrey-Kirwan residue in the formula is zero unless $\{v_i\}_{\beta_i\leq 0}$ lie in a cone on the boundary of C_Δ. Since β defines a relation among v_i, it follows that $\{v_i\}_{\beta_i\neq 0}$ lie in a proper face of C_Δ. Let C_0 be the minimal such face. By the same lemma, it now also follows that for the Jeffrey-Kirwan residue to be nonzero, $\{v_i\}_{i\in J}$ must lie in a face C_1 of C_Δ containing C_0.

If $V(J)\neq 0$ in the sum (1) above then $\{v_i\}_{i\in J}$ lie in at most one codimension 1 face C_1 of C. Let us fix C_1 and prove

$$\sum V(J)^2\gamma^J\langle x_0\frac{x^J}{x^{\beta+\mathbf{1}}}\rangle_{JK(\hat{\Sigma})}=0, \tag{2}$$

where the sum now runs over all $J\subset\{1,\ldots,n\}$, $|J|=d$ such that $\{v_i\}_{i\in J}$ lie in the face C_1. We get the sum (2) from (1) by formally setting $\gamma_i=0$ for $v_i\notin C_1$, hence going back to the determinantal form, we can write the sum (2) as

$$\langle\det(B)\frac{x_0}{x^{\beta+\mathbf{1}}}\rangle_{JK(\hat{\Sigma})},$$

where B is the matrix

$$B=(\sum_{v_k\in C_1}(w_i,v_k)(w_j,v_k)\gamma_k x_k)_{i,j=1,\ldots,d}.$$

Choose w_1 so that

$$(w_1,v_k)=\frac{1}{\gamma_k},\quad v_k\in C_1.$$

Such w_1 can be taken as a linear combination of w_γ and $w_1'\in N_K$ vanishing on C_1. Then the jth entry in the first row of B is

$$\sum_{v_k\in C_1}(w_j,v_k)x_k.$$

Since $\sum_{k=1}^n(w_j,v_k)x_k$ restricts to zero on $R(\hat{\Sigma})_K$, we may replace the jth entry by

$$-\sum_{v_k\notin C_1}(w_j,v_k)x_k.$$

After doing this replacement, Borisov [5] showed that the support of $\det B$ does not intersect any codimension 1 face of C_Δ containing C_0, hence the Jeffrey-Kirwan residue above is zero. Let us recall his argument.

Choose $w_2,\ldots,w_{r+1}$, where $r=d+1-\dim C_0$, so that they vanish on C_0. (This choice is made independent of the choice of C_1.) Suppose a monomial x^I that occurs in $\det B$ with nonzero coefficient is supported in a codimension 1 face C_1' of C_Δ containing C_0. Then we can write $I=\{i_1,\ldots,i_d\}$, where $v_{i_1}\in C_1'\smallsetminus C_1$ and $v_{i_2},\ldots,v_{i_d}\in C_1'\cap C_1$. Here x_{i_1} comes from the first row of the matrix B and $x_{i_2},\ldots,x_{i_d}$ from the rows $2,\ldots,d$. Because $C_1'\neq C_1$, a nontrivial linear combination of $w_2,\ldots,w_{r+1}$ vanishes on $C_1'\cap C_1$. It follows that $x_{i_2},\ldots,x_{i_d}$ do not occur in the nonzero minors of B constructed from rows $2,\ldots,r+1$. $\square$

Let $P(x_1,\dots,x_n)\in K[x_1,\dots,x_n]$ be a homogeneous polynomial of degree $d+1$ such that $P(a_1t^{v_1},\dots,a_nt^{v_n})\in I_\Delta$. It is known that the residue

$$Res_{a_1,\dots,a_n}P(a_1t^{v_1},\dots,a_nt^{v_n})$$

is a rational function in a_i with denominator the principal determinant E_f [8]. The support of E_f is the secondary polytope of Δ, with vertices corresponding to coherent triangulations of Δ. Consider the vertex corresponding to the triangulation $\mathcal{T}$ and expand $Res_{a_1,\dots,a_n}P(a_1t^{v_1},\dots,a_nt^{v_n})$ in a Laurent series at that vertex. Since the inner cone to the secondary polytope at the vertex corresponding to $\mathcal{T}$ is the cone $R(\Sigma)_{\mathrm{eff}}$, the expansion of the residue lies in the ring that we denoted $K[[a_1,\dots,a_n]]$. We claim that this expansion is precisely the one given by the residue mirror map RM. Indeed, modulo the ideal $(f_0,\dots,f_d)$, we can express

$$P(a_1t^{v_1},\dots,a_nt^{v_n})=\frac{g(a_1,\dots,a_n)}{E_f(a_1,\dots,a_n)}H_f,$$

for some polynomial $g(a_1,\dots,a_n)$. Then

$$E_fRes_{a_1,\dots,a_n}P(a_1t^{v_1},\dots,a_nt^{v_n})=gRes_{a_1,\dots,a_n}H_f=gVol(\Delta),$$

and the same formula holds if we replace $Res_{a_1,\dots,a_n}$ by RM. Since E_f has a unique inverse in $K[[a_1,\dots,a_n]]$, we get that the two Laurent series are equal. We state this as a theorem.

Theorem 3.3. *Let $P(x_1,\dots,x_n)\in K[x_1,\dots,x_n]$ be a homogeneous polynomial of degree $d+1$ such that $P(a_1t^{v_1},\dots,a_nt^{v_n})\in I_\Delta$. The Laurent series expansion of*

$$Res_{a_1,\dots,a_n}P(a_1t^{v_1},\dots,a_nt^{v_n})$$

at the vertex of the secondary polytope of Δ corresponding to the triangulation $\mathcal{T}$ is

$$Res_{a_1,\dots,a_n}P(a_1t^{v_1},\dots,a_nt^{v_n})=\sum_{\beta\in R(\Sigma)_{\mathrm{eff}}}\langle P(x_1,\dots,x_n)\frac{1}{x^{\beta+\mathbf{1}}}\rangle_{JK(\hat{\Sigma})}a^\beta.$$

□

In particular, the coefficient of a^β in the series above does not depend on the chosen completion $\hat{\Sigma}$ of the fan Σ.

4. Morrison-Plesser fans

Consider one coefficient of the series in Theorem 3.3:

$$\langle P(x_1,\dots,x_n)\frac{1}{x^{\beta+\mathbf{1}}}\rangle_{JK(\hat{\Sigma})}.$$

Our goal in this section is to construct a new complete fan $\hat{\Sigma}_\beta$, the Morrison-Plesser fan, such that the Jeffrey-Kirwan residue above can be identified with the evaluation map applied to a top degree cohomology class in $H(\hat{\Sigma}_\beta)$. In the next sections we apply this construction to other complete projective fans.

It turns out that $\hat{\Sigma}_\beta$ has a natural description in terms of Gale dual configurations [10]. We translate these dual notions into the more conventional setting of fans.

Let us start by recalling the construction of the fan $\hat{\Sigma}$ as a quotient, corresponding to the construction of a toric variety as a GIT quotient. First note that $\hat{\Sigma}$ is projective. One can extend a strictly convex conewise linear function on Σ to such a function L on $\hat{\Sigma}$ by choosing $L(v_0) \gg 0$. Consider the exact sequence

$$0 \to R(\hat{\Sigma}) \to \mathbb{Z}^{n+1} \xrightarrow{\hat{\pi}} M,$$

and fix an ample class $[L] \in H^1(\hat{\Sigma}) \simeq R(\hat{\Sigma})^*_K$. Then the pair $(\hat{\pi}, [L])$ determines the fan $\hat{\Sigma}$ completely as follows. The Gale dual of a cone $\sigma \subset M_K$ generated by $\{v_i\}_{i\in I}$ is the cone in $R(\hat{\Sigma})^*_K$ generated by the images of $\{e_i^*\}_{i\notin I}$ under the map $(\mathbb{Z}^{n+1})^* \to R(\hat{\Sigma})^*$. Then $\sigma \in \hat{\Sigma}$ if and only if its Gale dual contains $[L]$ in its interior. The completeness of the fan $\hat{\Sigma}$ corresponds to the condition that the images of $e_0^*, \ldots, e_n^*$ in $R(\hat{\Sigma})^*_K$ lie in an open half-space; $\hat{\Sigma}$ being simplicial is equivalent to the condition that $[L]$ does not lie in a smaller dimensional cone generated by the images of a subset of e_i^*. The Gale dual cones also determine the Jeffrey-Kirwan residue, and hence the evaluation map in the cohomology of $\hat{\Sigma}$. The volume of a cone $\sigma \in \hat{\Sigma}$ is equal to the volume of its Gale dual if $\hat{\pi}$ is surjective; otherwise the volumes differ by a constant factor, the index $[M : \hat{\pi}(\mathbb{Z}^{n+1})]$.

Let us fix $\beta \in \mathbb{Z}^{n+1}$ (take $\beta_0 = 0$ if $\beta \in R(\Sigma) \subset \mathbb{Z}^n$) and write $\beta = \beta^+ - \beta^-$, where $\beta_i^\pm = \max(\pm\beta_i, 0)$. Denote $|\beta^+| = \sum_{i=0}^n \beta_i^+$. The following construction of $\hat{\Sigma}_\beta$ only depends on β^+.

Let $\rho : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1+|\beta^+|}$ be the product of diagonal embeddings $\mathbb{Z} \to \mathbb{Z}^{1+\beta_i^+}$ for $i = 0, \ldots, n$. Define M_β as the pushout of ρ and $\hat{\pi}$:

$$\begin{array}{ccc} \mathbb{Z}^{n+1+|\beta^+|} & \longrightarrow & M_\beta \\ \uparrow \rho & & \uparrow \\ \mathbb{Z}^{n+1} & \xrightarrow{\hat{\pi}} & M. \end{array}$$

In other words,

$$M_\beta = (\mathbb{Z}^{n+1+|\beta^+|} \times M)/\mathbb{Z}^{n+1},$$

where $\mathbb{Z}^{n+1}$ is mapped to the product diagonally. Since ρ embeds $\mathbb{Z}^{n+1}$ in $\mathbb{Z}^{n+1+|\beta^+|}$ as a direct summand, M_β has no torsion. From the pushout diagram we also get an exact sequence

$$0 \to R(\hat{\Sigma}) \to \mathbb{Z}^{n+1+|\beta^+|} \xrightarrow{\hat{\pi}_\beta} M_\beta$$

and an isomorphism between the cokernels of $\hat{\pi}$ and $\hat{\pi}_\beta$. Let $\hat{\Sigma}_\beta$ be the fan defined by the pair $(\hat{\pi}_\beta, [L])$.

We denote the basis of $\mathbb{Z}^{n+1+|\beta^+|}$ by $\{e_{i,j}\}_{i=0,\ldots,n; j=0,\ldots,\beta_i^+}$ and the corresponding generators of the fan $\hat{\Sigma}_\beta$ by $v_{i,j} \in M_\beta$. The images of the dual basis elements $e_{i,j}^*$ under $(\mathbb{Z}^{n+1+|\beta^+|})^* \to R(\hat{\Sigma})^*$ coincide with the images of e_i^* under $(\mathbb{Z}^{n+1})^* \to R(\hat{\Sigma})^*$. It follows from this that $\hat{\Sigma}_\beta$ is complete and simplicial. Moreover, the Jeffrey-Kirwan residue in the fan $\hat{\Sigma}_\beta$ of a rational function in the variables $x_{i,j}$ is equal to the Jeffrey-Kirwan residue in the fan $\hat{\Sigma}$ of the same function but with $x_{i,j}$ replaced by x_i:

$$\langle f(x_{i,j}) \rangle_{JK(\hat{\Sigma}_\beta)} = \langle f(x_i) \rangle_{JK(\hat{\Sigma})}.$$

Now consider the Jeffrey-Kirwan residue at the beginning of this section. We can express it as:

$$\langle P(x_1,\ldots,x_n)\frac{1}{x^{\beta+\mathbf{1}}}\rangle_{JK(\hat{\Sigma})} = \langle\frac{P(x_{1,0},\ldots,x_{n,0})x_{1,0}^{\beta_1^-}\cdots x_{n,0}^{\beta_n^-}}{\prod_{i,j}x_{i,j}}\rangle_{JK(\hat{\Sigma}_\beta)}$$
$$= \langle P(\chi_{1,0},\ldots,\chi_{n,0})\chi_{1,0}^{\beta_1^-}\cdots\chi_{n,0}^{\beta_n^-}\rangle_{\hat{\Sigma}_\beta},$$

where we have denoted by $\chi_{i,j}$ the generators of the cohomology of $\hat{\Sigma}_\beta$ corresponding to $v_{i,j}$. Let us call

$$\Phi_\beta = [\chi_{1,0}^{\beta_1^-}\cdots\chi_{n,0}^{\beta_n^-}] \in H(\hat{\Sigma}_\beta)$$

the Morrison-Plesser class. Then we have:

Theorem 4.1. *Let $P(x_1,\ldots,x_n)\in K[x_1,\ldots,x_n]$ be a homogeneous polynomial of degree $d+1$ such that $P(a_1t^{v_1},\ldots,a_nt^{v_n})\in I_\Delta$. The Laurent series expansion of*

$$Res_{a_1,\ldots,a_n}P(a_1t^{v_1},\ldots,a_nt^{v_n})$$

at the vertex of the secondary polytope of Δ corresponding to the triangulation $\mathcal{T}$ is

$$Res_{a_1,\ldots,a_n}P(a_1t^{v_1},\ldots,a_nt^{v_n}) = \sum_{\beta\in R(\Sigma)_{\text{eff}}}\langle P(\chi_{1,0},\ldots,\chi_{n,0})\Phi_\beta\rangle_{\hat{\Sigma}_\beta}a^\beta.$$

□

Remark 4.2. Let Σ_β be the fan obtained from $\hat{\Sigma}_\beta$ by removing the ray generated by $v_{0,0}$ and all cones containing it. Similarly to Σ, the fan Σ_β is a subdivision of a pointed cone in $M_{\beta,K}$. The fan Σ_β does not depend on the completion $\hat{\Sigma}$ and it can be constructed directly from Σ by a construction similar to $\hat{\Sigma}_\beta$. It is also possible to show (considering $\Phi_\beta\in\mathcal{A}(\Sigma_\beta)$):

$$P(\chi_{1,0},\ldots,\chi_{n,0})\Phi_\beta\in\mathcal{A}(\Sigma_\beta,\partial\Sigma_\beta),$$

hence we can write the series in Theorem 4.1 as

$$Res_{a_1,\ldots,a_n}P(a_1t^{v_1},\ldots,a_nt^{v_n}) = \sum_{\beta\in R(\Sigma)_{\text{eff}}}\langle P(\chi_{1,0},\ldots,\chi_{n,0})\Phi_\beta\rangle_{\Sigma_\beta}a^\beta.$$

This gives an expansion of the residue independent from the completion $\hat{\Sigma}$. However, neither $P(\chi_{1,0},\ldots,\chi_{n,0})$ nor Φ_β may vanish on $\partial\Sigma_\beta$, hence we can not consider Φ_β as an element in $H(\Sigma_\beta)$ or $H(\Sigma_\beta,\partial\Sigma_\beta)$.

5. Calabi-Yau hypersurfaces

In this section we explain how the toric residue mirror conjecture for Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties [2, 5, 10] follows from Theorem 3.3.

Assume that the polytope Δ is reflexive; that means, its polar is also a lattice polytope. Then $0\in\Delta$ is the unique lattice point in the interior of Δ. We assume that 0 is a vertex of every maximal simplex in $\mathcal{T}$. Let the generators of the fan Σ in $M=\overline{M}\times\mathbb{Z}$ be $v_i=(\bar{v}_i,1)$ for $i=1,\ldots,n$ and $v_{n+1}=(0,1)$. Also let $q: M\to\overline{M}$ be the projection. Then q maps the fan Σ to a complete fan $\overline{\Sigma}$ and we have isomorphisms:

$$H^i(\overline{\Sigma})\xrightarrow{q^*}H^i(\Sigma)\xrightarrow{\chi_{n+1}}H^{i+1}(\Sigma,\partial\Sigma).$$

These isomorphisms are compatible with evaluation: if $P(x_1,\ldots,x_n)$ is a homogeneous polynomial of degree d then

$$\langle P(\bar{\chi}_1,\ldots,\bar{\chi}_n)\rangle_{\overline{\Sigma}} = \langle \chi_{n+1} P(\chi_1,\ldots,\chi_n)\rangle_{\Sigma},$$

where $\bar{\chi}_i$ are the generators of the cohomology of $\overline{\Sigma}$ corresponding to $\bar{v}_i$. We wish to give a similar correspondence between the Jeffrey-Kirwan residues in $\overline{\Sigma}$ and $\hat{\Sigma}$.

Let us choose the completion $\hat{\Sigma}$ by taking $v_0=(0,-1)$, and consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & R(\hat{\Sigma}) & \to & \mathbb{Z}^{n+2} & \to & M \\ & & \downarrow & & \downarrow p & & \downarrow q \\ 0 & \to & R(\overline{\Sigma}) & \to & \mathbb{Z}^{n} & \to & \overline{M}, \end{array}$$

where the middle vertical map is defined by $p(e_i)=e_i$ for $i=1,\ldots,n$ and $p(e_0)=p(e_{n+1})=0$. It follows that functions defined on $R(\hat{\Sigma})_K$ by x_i for $i=1,\ldots,n$ are the pullbacks of functions defined by x_i on $R(\overline{\Sigma})_K$. The hyperplanes defined by $x_0=0$ and $x_{n+1}=0$ map onto $R(\overline{\Sigma})_K$. Comparing the volumes of cones in $\overline{\Sigma}$ and in $\hat{\Sigma}$, we get

$$\langle x^m\rangle_{JK(\overline{\Sigma})} = \langle x^m \frac{1}{x_0}\rangle_{JK(\hat{\Sigma})}$$

for any Laurent monomial $x^m\in K[x_1^{\pm1},\ldots,x_n^{\pm1}]$. If $l>0$ then

$$\langle x_0^l x^m \frac{1}{x_0}\rangle_{JK(\hat{\Sigma})} = 0$$

because all linear forms in the denominator are pulled back from $R(\overline{\Sigma})_K^*$, hence they do not span $R(\hat{\Sigma})_K^*$. Using the linear relation $-x_0+x_1+\ldots+x_{n+1}=0$ on $R(\hat{\Sigma})_K$, we get for $k\geq 0$

$$\begin{aligned} \langle x^m x_{n+1}^k \frac{1}{x_0}\rangle_{JK(\hat{\Sigma})} &= \langle x^m (x_0-x_1-\ldots-x_n)^k \frac{1}{x_0}\rangle_{JK(\hat{\Sigma})} \\ &= \langle x^m (-x_1-\ldots-x_n)^k \frac{1}{x_0}\rangle_{JK(\hat{\Sigma})} \\ &= \langle x^m (-x_1-\ldots-x_n)^k \rangle_{JK(\overline{\Sigma})}. \end{aligned}$$

Let $R(\overline{\Sigma})$ be the group of relations among $\bar{v}_i$. We have an isomorphism

$$\begin{gathered} R(\Sigma)\to R(\overline{\Sigma}) \\ (\beta_1,\ldots,\beta_{n+1})\mapsto(\beta_1,\ldots,\beta_n), \end{gathered}$$

with inverse defined by $\beta_{n+1}=-\beta_1-\ldots-\beta_n$. The dual map $H^1(\overline{\Sigma})\to H^1(\Sigma)$ identifies the ample cones of the two fans, hence the map above identifies the Mori cones. Note also that if $\beta\in R(\Sigma)_{\text{eff}}$ then $\beta_{n+1}\leq 0$ because $-\chi_{n+1}$ is convex and so it lies in the ample cone of Σ.

For $\beta\in R(\overline{\Sigma})_{\text{eff}}$, let $\overline{\Sigma}_\beta$ be the Morrison-Plesser fan constructed from $\overline{\Sigma}$. Define the Morrison-Plesser class $\Phi_\beta\in H(\overline{\Sigma}_\beta)$:

$$\Phi_\beta = \bar{\chi}^{\beta^-} = \bar{\chi}_{1,0}^{\beta_1^-}\cdots\bar{\chi}_{n,0}^{\beta_n^-}(-\bar{\chi}_{1,0}-\ldots-\bar{\chi}_{n,0})^{\beta_1+\ldots+\beta_n},$$

where $\bar{\chi}_{i,j}$ are the generators of the cohomology of $\overline{\Sigma}_\beta$ corresponding to $\bar{v}_{i,j}$.

The ideal $I_\Delta \subset S_\Delta$ is principal, generated by $t^{v_{n+1}}$. Consider one coefficient in the series of Theorem 3.3 applied to the polynomial $a_{n+1}t^{v_{n+1}}P(a_1t^{v_1},\ldots,a_nt^{v_n}) \in I_\Delta$:

$$\begin{aligned}\langle x_{n+1}P(x_1,\ldots,x_n)\frac{1}{x^{\beta+\mathbf{1}}}\rangle_{JK(\hat{\Sigma})} &= \langle P(x_1,\ldots,x_n)\frac{x^{\beta^-}}{x^{\beta^+}}\frac{1}{x_0x_1\cdots x_n}\rangle_{JK(\hat{\Sigma})}\\ &= \langle P(x_1,\ldots,x_n)\frac{x^{\beta^-}}{x^{\beta^+}}\frac{1}{x_1\cdots x_n}\rangle_{JK(\overline{\Sigma})}\\ &= \langle P(\bar{\chi}_{1,0},\ldots,\bar{\chi}_{n,0})\Phi_\beta\rangle_{\overline{\Sigma}_\beta}.\end{aligned}$$

Thus, we get:

Theorem 5.1. *Let $P(x_1,\ldots,x_n) \in K[x_1,\ldots,x_n]$ be a homogeneous polynomial of degree d. The Laurent series expansion of $Res_{a_1,\ldots,a_n,a_{n+1}=1}(t^{v_{n+1}}P(a_1t^{v_1},\ldots,a_nt^{v_n}))$ at the vertex of the secondary polytope of Δ corresponding to the triangulation $\mathcal{T}$ is*

$$Res_{a_1,\ldots,a_n}(t^{v_{n+1}}P(a_1t^{v_1},\ldots,a_nt^{v_n})) = \sum_{\beta\in R(\overline{\Sigma})_{\text{eff}}} \langle P(\bar{\chi}_{1,0},\ldots,\bar{\chi}_{n,0})\Phi_\beta\rangle_{\overline{\Sigma}_\beta}a^\beta.$$

□

In [2] the parameters a_i differ by a sign from the ones used here. This introduces a sign difference in the definition of Φ_β and in the Laurent series expansion.

6. Complete intersections

In this section we prove the toric residue mirror conjecture for Calabi-Yau complete intersections in Gorenstein toric Fano varieties [3]. The construction relies on the Cayley trick [3] and the proof is completely analogous to the hypersurface case.

Let $\Delta \in \overline{M}_K$ be a reflexive polytope ($\Delta = \nabla^*$ in [3]), and $\mathcal{T}$ a coherent triangulation of Δ such that $0 \in \Delta$ is a vertex of every maximal simplex. Let $\overline{\Sigma}$ be the complete simplicial fan in $\overline{M}_K$ defined by $\mathcal{T}$. Denote by $\{\bar{v}_1,\ldots,\bar{v}_n\}$ the primitive generators of $\overline{\Sigma}$, and let L be the conewise linear function on $\overline{\Sigma}$ such that $L(\bar{v}_i) = 1$ for $i = 1,\ldots,n$. A *nef-partition* [4] of L is an expression

$$L = l_1 + l_2 + \ldots + l_r,$$

where l_i are integral non-negative convex conewise linear functions on $\overline{\Sigma}$. We assume that all $l_i \neq 0$. A nef-partition defines a partition of $\{1,\ldots,n\}$ into a disjoint union $E_1 \cup \ldots \cup E_r$, where $E_j = \{i | l_j(v_i) = 1\}$. Let

$$\Delta_j = conv(\{0\} \cup \{v_i\}_{i\in E_j}).$$

Let $M = \overline{M} \times \mathbb{Z}^r$. Define the Cayley polytope

$$\tilde{\Delta} = \Delta_1 * \cdots * \Delta_r = conv(\Delta_1 \times \{(0,e_1)\} \cup \ldots \cup \Delta_r \times \{(0,e_r)\}),$$

where $e_1,\ldots,e_r$ is the standard basis of $\mathbb{Z}^r$, and let $C_{\tilde{\Delta}}$ be the cone over $\tilde{\Delta}$. The lattice points in $\tilde{\Delta}$ are $v_i = (\bar{v}_i, e_j)$ for $i = 1,\ldots,n$, where $i \in E_j$ and $v_{n+j} = (0, e_j)$ for $j = 1,\ldots,r$. The triangulation $\mathcal{T}$ defines a triangulation $\tilde{\mathcal{T}}$ of $\tilde{\Delta}$, hence a simplicial subdivision of the cone $C_{\tilde{\Delta}}$ into a fan Σ as follows. Let the maximal cones of Σ be generated by

$$\{v_{n+1},\ldots,v_{n+r}\} \cup \{v_i\}_{\bar{v}_i\in\sigma}$$

for some maximal cone $\sigma \in \overline{\Sigma}$.

Let $q : M \to \overline{M}$ be the projection, mapping the fan Σ to the fan $\overline{\Sigma}$. Since every maximal cone in Σ is the product of a cone in $\overline{\Sigma}$ with the simplicial cone generated by $\{v_{n+1}, \ldots, v_{n+r}\}$, we get isomorphisms

$$H^i(\overline{\Sigma}) \xrightarrow{q^*} H^i(\Sigma) \xrightarrow{\chi_{n+1}\cdots\chi_{n+r}} H^{i+r}(\Sigma, \partial\Sigma).$$

These isomorphisms are compatible with evaluation maps: if $P(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree d then

$$\langle P(\bar{\chi}_1, \ldots, \bar{\chi}_n)\rangle_{\overline{\Sigma}} = \langle \chi_{n+1}\cdots\chi_{n+r} P(\chi_1, \ldots, \chi_n)\rangle_{\Sigma}.$$

We complete Σ to $\hat{\Sigma}$ by adding the ray generated by $v_0 = -v_{n+1} - \ldots - v_{n+r}$ and consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \to & R(\hat{\Sigma}) & \to & \mathbb{Z}^{n+r+1} & \to & M \\
 & & \downarrow & & \downarrow p & & \downarrow q \\
0 & \to & R(\overline{\Sigma}) & \to & \mathbb{Z}^n & \to & \overline{M},
\end{array}$$

where the middle vertical map is defined by $p(e_i) = e_i$ for $i = 1, \ldots, n$ and $p(e_i) = 0$ for $i = 0, n+1, \ldots, n+r$. The functions defined on $R(\hat{\Sigma})_K$ by x_i for $i = 1, \ldots, n$ are pullbacks of functions on $R(\overline{\Sigma})_K$; the hyperplanes defined by $x_i = 0$ for $i = 0, n+1, \ldots, n+r$ map onto $R(\overline{\Sigma})_K$. One easily checks (for example, using the comparison of the evaluation maps in $H(\overline{\Sigma})$ and $H(\hat{\Sigma})$) that

$$\langle x^m \rangle_{JK(\overline{\Sigma})} = \langle x^m \frac{1}{x_0} \rangle_{JK(\hat{\Sigma})}$$

for any Laurent monomial $x^m \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. As in the previous section, we also have

$$\langle x_0^l x^m \frac{1}{x_0} \rangle_{JK(\hat{\Sigma})} = 0$$

for any $l > 0$, and using the relations $-x_0 + x_{n+j} + \sum_{i \in E_j} x_i = 0$ on $R(\hat{\Sigma})_K$, we get for $k_i \geq 0$

$$\langle x^m x_{n+1}^{k_1} \cdots x_{n+r}^{k_r} \rangle_{JK(\hat{\Sigma})} = \langle x^m (-\sum_{i \in E_1} x_i)^{k_1} \cdots (-\sum_{i \in E_r} x_i)^{k_r} \rangle_{JK(\overline{\Sigma})}.$$

Forgetting the last r coordinates of vectors in $\mathbb{Z}^{n+r}$, we get an isomorphism

$$R(\Sigma) \to R(\overline{\Sigma})$$
$$(\beta_1, \ldots, \beta_{n+r}) \mapsto (\beta_1, \ldots, \beta_n),$$

with inverse defined by $\beta_{n+j} = -\sum_{i \in E_j} \beta_i$. This isomorphism identifies the Mori cones of Σ and $\overline{\Sigma}$. If $\beta \in R(\Sigma)_{\text{eff}}$ then $\beta_{n+j} \leq 0$ because $-\chi_{n+j}$ lies in the ample cone of Σ.

For $\beta \in R(\overline{\Sigma})_{\text{eff}}$, let $\overline{\Sigma}_\beta$ be the Morrison-Plesser fan constructed from $\overline{\Sigma}$. Define the Morrison-Plesser class $\Phi_\beta \in H(\overline{\Sigma}_\beta)$:

$$\Phi_\beta = \bar{\chi}^{\beta^-} = \bar{\chi}_{1,0}^{\beta_1^-} \cdots \bar{\chi}_{n,0}^{\beta_n^-} (-\sum_{i \in E_1} \bar{\chi}_{i,0})^{\sum_{i \in E_1} \beta_i} \cdots (-\sum_{i \in E_r} \bar{\chi}_{i,0})^{\sum_{i \in E_r} \beta_i}.$$

The ideal $I_{\tilde{\Delta}} \subset S_{\tilde{\Delta}}$ is again principal, generated by $t^{v_{n+1}} \cdots t^{v_{n+r}}$. Thus, we have

Theorem 6.1. *Let $P(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. The Laurent series expansion of*

$$Res_{a_1,\ldots,a_n,a_{n+1}=\ldots=a_{n+r}=1}(t^{v_{n+1}} \cdots t^{v_{n+r}} P(a_1 t^{v_1}, \ldots, a_n t^{v_n}))$$

at the vertex of the secondary polytope of $\tilde{\Delta}$ corresponding to the triangulation $\tilde{\mathcal{T}}$ is

$$Res_{a_1,\ldots,a_n}(t^{v_{n+1}} \cdots t^{v_{n+r}} P(a_1 t^{v_1}, \ldots, a_n t^{v_n})) = \sum_{\beta \in R(\overline{\Sigma})_{\text{eff}}} \langle P(\bar{\chi}_{1,0}, \ldots, \bar{\chi}_{n,0}) \Phi_\beta \rangle_{\overline{\Sigma}_\beta} a^\beta.$$

□

7. Mixed residues and mixed volumes

We keep the notation from the previous section.

The ring $S_{\tilde{\Delta}}$ is graded by $\mathbb{Z}^r_{\geq 0}$ and $I_{\tilde{\Delta}} \subset S_{\tilde{\Delta}}$ is a homogeneous ideal. For a partition $k = (k_1, \ldots, k_r)$,

$$k_1 + \ldots + k_r = n + d, \qquad k_i > 0,$$

the restriction of $Res_{a_1,\ldots,a_n}$ to the degree k component of I_Δ is called the k-mixed residue.

The following was conjectured by Batyrev and Materov [3]:

Theorem 7.1. *Let H_f^k be the k-homogeneous component of H_f. The k-mixed residue of H_f^k is*

$$Res_{a_1,\ldots,a_n} H_f^k = V(\Delta_1^{\bar{k}_1} \cdots \Delta_r^{\bar{k}_r}),$$

where the right hand side denotes the mixed volume multiplied with $(n + d - 1)!$, and $\bar{k} = (k_1 - 1, \ldots, k_r - 1)$.

Proof. Let $c_1, \ldots, c_k$ be parameters close to 1 and consider the (non-integral) polytope

$$\tilde{\Delta}_c = c_1 \Delta_1 * \ldots * c_r \Delta_r.$$

The volume of $\tilde{\Delta}_c$ is a polynomial in c_i with coefficients the normalized mixed volumes:

$$Vol(\tilde{\Delta}_c) = \sum_k V(\Delta_1^{\bar{k}_1} \cdots \Delta_r^{\bar{k}_r}) c_1^{\bar{k}_1} \cdots c_r^{\bar{k}_r}.$$

We may take this as the definition of the mixed volume.

The triangulation $\tilde{\mathcal{T}}$ of $\tilde{\Delta}$ induces a triangulation $\tilde{\mathcal{T}}_c$ of $\tilde{\Delta}_c$ if we replace the vertices $v_i = (\bar{v}_i, e_j)$ by $v_{i,c} = (c_j \bar{v}_i, e_j)$, and leave $v_{n+j,c} = v_{n+j} = (0, e_j)$. If a simplex $\tau \in \tilde{\mathcal{T}}$ corresponds to the simplex $\tau_c \in \tilde{\mathcal{T}}_c$, then an easy determinant computation shows that

$$Vol(\tau_c) = Vol(\tau) c_1^{\bar{k}_1} \cdots c_r^{\bar{k}_r},$$

where $\bar{k}_j = |\{i \in E_j | v_i \in \tau\}|$.

Let us write $\gamma = (\gamma_1, \ldots, \gamma_{n+r})$, where $\gamma_i = c_j$ if $i \in E_j$ or if $i = n + j$. We apply Lemma 3.2 to get:

$$\begin{aligned} H_{f_\gamma} = \sum_k H_f^k c_1^{k_1} \cdots c_r^{k_r} &\overset{RM}{\longmapsto} \sum_{\sigma\in\Sigma} Vol(\sigma) \prod_{v_i\in\sigma} \gamma_i \\ &= \sum_{\sigma\in\Sigma} Vol(\sigma_c) c_1 \cdots c_r \\ &= Vol(\tilde{\Delta}_c) c_1 \cdots c_r \\ &= \sum_k V(\Delta_1^{\bar{k}_1} \cdots \Delta_r^{\bar{k}_r}) c_1^{k_1} \cdots c_r^{k_r}. \end{aligned}$$

Comparing the coefficents on both sides we get the desired result. □

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